# SHORTER ARTICLES AND DISCUSSION

## A NOTE ON CERTAIN BIOMETRICAL COMPUTATIONS[1]

1. It is a well known fact that curves of individual growth, in which some size character of the organism is taken as ordinate, and time as abscissa, are closely related to a logarithmic curve. To Pearson[2] belongs the credit of first demonstrating this concretely by fitting a logarithmic curve to growth data. Since that time a number of other students[3] of growth have made use of such curves in graduating observational data.

Now while the simplest logarithmic curve

$$y = a + b \log x \qquad \text{(i)}$$

is probably only very exceptionally (if ever) followed precisely in the growth of an organism, yet it certainly represents the general type towards which many observational growth curves tend. In the practical analysis of growth data it is often found to be extremely helpful as the first step to fit such a curve as (i) or a simple variant of it in which a "line" term is added, as in

$$y = a + bx + c \log x. \qquad \text{(ii)}$$

Actually finding out by trial just wherein a curve like (ii) fails to fit the data—if it does fail—will usually give one the clue as to the way in which the curve must be modified in order to graduate the observations satisfactorily.

In fitting a curve like (ii) to a series of observations by the method of least squares the type equations are as follows:

$$\left.\begin{aligned} &S(y) - na - bS(x) - cS(\log x) = 0, \\ &S(xy) - aS(x) - bS(x^2) - cS(x \log x) = 0, \\ &S(y \log x) - aS(\log x) - bS(x \log x) - cS(\log x)^2 = 0, \end{aligned}\right\} \text{(iii)}$$

where $S$ denotes summation for the $n$ values of the variables.

Now it is evident that, of the 11 summations included in these equations, only 3 involve the variable $y$. All the others are func-

[1] Papers from the Biological Laboratory of the Maine Agricultural Experiment Station. No. 31.

[2] Pearson, K., *Biometrika,* IV, 131–190. Cf. also Lewenz, M. A., and Pearson, K., *ibid.,* III, 367–397.

[3] Cf., for example, Pearl, R., Pepper, O. M., and Hagle, F. J., Carnegie Institution Publ. No. 58, 1907, and Donaldson, H. H., in *Jour. Comp. Neurol. and Psychol.,* XVIII, 345–392, 1908, and also in later papers.

tions of $x$. In practise many cases arise where all the base elements of the observational curve are *equal* and the values of $x$ run in ordinal units from 1 to whatever number the observations comprise. In such cases, taking the origin of $x$ at 0, the sums in (iii) which involve $x$ and $x^2$ may be read off at once from Elderton's[4] tables of the sums of the powers of the natural numbers. If, now, similar tables are available from which one can obtain the values of $S(\log x)$, $S(x \log x)$ and $S(\log x)^2$ for integral values of $x$, there are left only the three sums in which $y$ is involved which must be directly calculated.

So far as we are aware no tables have hitherto been published giving the sums of these logarithmic functions of the natural numbers. Consequently the present short table has been prepared. The immediate incentive to its calculation was the fact that in studies on growth and related topics in this laboratory it has been rather frequently necessary to fit these simple logarithmic curves. The table was calculated several years ago purely as a labor saving factor in the work of the laboratory. It has been used in manuscript here since that time. It seems desirable to publish it in order that other workers may have the benefit of the time and effort which it saves in curve-fitting work of this sort.

2. The values of $S(\log x)$, $S(x \log x)$ and $S(\log x)^2$ given in the appended table were calculated twice independently, once with 10-place values of the logarithms, and once with 7-place figures. The 10-place logarithms were taken from Vega's Thesaurus,[5] and the multiplications and summations were performed on a large size Brunsviga arithmometer. As was to be expected, the values of $S(x \log x)$ and $S(\log x)^2$ for the higher numbers, when calculated from 7-place logarithms, were not accurate beyond the fifth place. This 7-place table merely served as a rough check on the accuracy of the 10-place work. The tabled values given in this paper were all obtained by cutting off the last 3 figures from the values in the 10-place table. The accuracy of these last figures was previously tested by differences. The table as given is believed to be accurate in the seventh place. This is entirely sufficient because, as a matter of fact, in practical curve fitting work one will not ordinarily use more than 4 or at the most 5 places of figures in the logarithms.

[4] *Biometrika*, II, 474–480.

[5] For the loan of a copy of these tables we are greatly indebted to Dr. H. G. Kribs, of the University of Pennsylvania.

3. The use of the tables may be illustrated from a concrete example based on data collected in this laboratory. Each of the successively laid eggs of a certain hen were measured, length and breadth being recorded. From these records the length-breadth index (100 breadth $\div$ length) was calculated. In all 87 eggs were measured.[6] To the line given by plotting the value of the index of each of these eggs in consecutive order as they were laid a curve of the type

$$y = a + bx + c(\log x)$$

was fitted by the method of least squares. In this equation $y$ denotes the value of the length-breadth index of an egg whose ordinal number in the whole series laid is $x$. That is, $S(x)$ will be the sum of the integers from 1 to 87 inclusive.

The type equations for this curve have been given above (p. 756) and need not be repeated. For the data under discussion $n = 87$. From the table given in the present paper we read off at once

$$S \log x = 132.3238,$$
$$S(x \log x) = 6{,}602.9556,$$
$$S(\log x)^2 = 215.0293.$$

Further from Elderton's table (*loc. cit.*) we get

$$S(x) = 3{,}828,$$
$$S(x^2) = 223{,}300.$$

This leaves to be obtained by actual addition from the data *only*

$$S(y) = 5{,}473.81,$$
$$S(xy) = 245{,}041.55,$$
$$S(y \log x) = 8{,}416.4497.$$

Substituting these values in the type equation (iii) we have

$$87a + 3{,}828b + 132.3238c = 5{,}473.81,$$
$$3{,}828a + 223{,}300b + 6{,}602.9556c = 245{,}041.55,$$
$$132.3238a + 6{,}602.9556b + 215.0293c = 8{,}416.4497.$$

Solving

$$y = 49.0241 - .0910x + 11.7669 \log x.$$

The goodness of fit of this curve may be judged by examination of Plate II of the paper where the original data are published.[7]

[6] The actual measurements of these eggs are given in detail in the *Journal of Experimental Zoology*, VI, 349.

[7] *Jour. Exper. Zool., loc. cit.*

TABLE OF THE SUMS OF THE LOGARITHMS OF THE NATURAL NUMBERS FROM 1 TO 100

| $x$ | $S(\log x)$ | $S(x \log x)$ | $S(\log x)^2$ |
|---|---|---|---|
| 1 | 0.0000000 | 0.0000000 | 0.0000000 |
| 2 | 0.3010300 | 0.6020600 | 0.0906191 |
| 3 | 0.7781513 | 2.0334238 | 0.3182638 |
| 4 | 1.3802112 | 4.4416637 | 0.6807400 |
| 5 | 2.0791812 | 7.9365137 | 1.1692991 |
| 6 | 2.8573325 | 12.6054212 | 1.7748184 |
| 7 | 3.7024305 | 18.5211075 | 2.4890091 |
| 8 | 4.6055205 | 25.7458274 | 3.3045806 |
| 9 | 5.5597630 | 34.3340100 | 4.2151594 |
| 10 | 6.5597630 | 44.3340100 | 5.2151594 |
| 11 | 7.6011557 | 55.7893295 | 6.2996581 |
| 12 | 8.6803370 | 68.7395045 | 7.4642903 |
| 13 | 9.7942803 | 83.2207681 | 8.7051601 |
| 14 | 10.9404084 | 99.2665606 | 10.0187696 |
| 15 | 12.1164996 | 116.9079295 | 11.4019602 |
| 16 | 13.3206196 | 136.1738492 | 12.8518651 |
| 17 | 14.5510685 | 157.0914808 | 14.3658697 |
| 18 | 15.8063410 | 179.6863859 | 15.9415788 |
| 19 | 17.0850946 | 203.9827044 | 17.5767895 |
| 20 | 18.3861246 | 230.0033043 | 19.2694686 |
| 21 | 19.7083439 | 257.7699095 | 21.0177324 |
| 22 | 21.0507666 | 287.3032084 | 22.8198311 |
| 23 | 22.4124944 | 318.6229487 | 24.6741338 |
| 24 | 23.7927057 | 351.7480185 | 26.5791169 |
| 25 | 25.1906457 | 386.6965187 | 28.5333531 |
| 26 | 26.6056190 | 423.4858257 | 30.5355027 |
| 27 | 28.0369828 | 462.1326474 | 32.5843049 |
| 28 | 29.4841408 | 502.6530722 | 34.6785713 |
| 29 | 30.9465388 | 545.0626142 | 36.8171792 |
| 30 | 32.4236601 | 589.3762518 | 38.9990664 |
| 31 | 33.9150218 | 635.6084643 | 41.2232261 |
| 32 | 35.4201717 | 683.7732636 | 43.4887026 |
| 33 | 36.9386857 | 733.8842237 | 45.7945871 |
| 34 | 38.4701646 | 785.9545068 | 48.1400148 |
| 35 | 40.0142326 | 839.9968884 | 50.5241609 |
| 36 | 41.5705351 | 896.0237784 | 52.9462384 |
| 37 | 43.1387369 | 954.0472422 | 55.4054951 |
| 38 | 44.7185205 | 1,014.0790189 | 57.9012113 |
| 39 | 46.3095851 | 1,076.1305385 | 60.4326979 |
| 40 | 47.9116451 | 1,140.2129382 | 62.9992941 |
| 41 | 49.5244289 | 1,206.3370763 | 65.6003659 |
| 42 | 51.1476782 | 1,274.5135465 | 68.2353041 |
| 43 | 52.7811467 | 1,344.7526901 | 70.9035233 |
| 44 | 54.4245993 | 1,417.0646079 | 73.6044600 |
| 45 | 56.0778119 | 1,491.4591710 | 76.3375716 |
| 46 | 57.7405697 | 1,567.9460313 | 79.1023352 |
| 47 | 59.4126676 | 1,646.5346306 | 81.8982465 |
| 48 | 61.0939088 | 1,727.2342100 | 84.7248186 |
| 49 | 62.7841049 | 1,810.0538179 | 87.5815814 |
| 50 | 64.4830749 | 1,895.0023181 | 90.4680804 |

| $x$ | $S$ (log $x$) | $S$ ($x$ log $x$) | $S$ (log $x$)$^2$ |
|---|---|---|---|
| 51 | 66.1906450 | 1,982.0883971 | 93.3838763 |
| 52 | 67.9066484 | 2,071.3205710 | 96.3285438 |
| 53 | 69.6309243 | 2,162.7071920 | 99.3016711 |
| 54 | 71.3633180 | 2,256.2564551 | 102.3028592 |
| 55 | 73.1036807 | 2,351.9764030 | 105.3317215 |
| 56 | 74.8518687 | 2,449.8749325 | 108.3878829 |
| 57 | 76.6077436 | 2,549.9597993 | 111.4709794 |
| 58 | 78.3711716 | 2,652.2386229 | 114.5806577 |
| 59 | 80.1420236 | 2,756.7188916 | 117.7165745 |
| 60 | 81.9201748 | 2,863.4079666 | 120.8783964 |
| 61 | 83.7055047 | 2,972.3130866 | 124.0657990 |
| 62 | 85.4978964 | 3,083.4413713 | 127.2784670 |
| 63 | 87.2972369 | 3,196.7998259 | 130.5160934 |
| 64 | 89.1034169 | 3,312.3953443 | 133.7783793 |
| 65 | 90.9163303 | 3,430.2347124 | 137.0650341 |
| 66 | 92.7358742 | 3,550.3246122 | 140.3757742 |
| 67 | 94.5619490 | 3,672.6716240 | 143.7103234 |
| 68 | 96.3944579 | 3,797.2822300 | 147.0684123 |
| 69 | 98.2333070 | 3,924.1628173 | 150.4497786 |
| 70 | 100.0784050 | 4,053.3196801 | 153.8541654 |
| 71 | 101.9296634 | 4,184.7590228 | 157.2813229 |
| 72 | 103.7869959 | 4,318.4869626 | 160.7310069 |
| 73 | 105.6503187 | 4,454.5095314 | 164.2029790 |
| 74 | 107.5195505 | 4,592.8326786 | 167.6970062 |
| 75 | 109.3946117 | 4,733.4622734 | 171.2128609 |
| 76 | 111.2754253 | 4,876.4041064 | 174.7503207 |
| 77 | 113.1619160 | 5,021.6638922 | 178.3091670 |
| 78 | 115.0540106 | 5,169.2472713 | 181.8891890 |
| 79 | 116.9516377 | 5,319.1598115 | 185.4901776 |
| 80 | 118.8547277 | 5,471.4070104 | 189.1119291 |
| 81 | 120.7632127 | 5,625.9942970 | 192.7542442 |
| 82 | 122.6770266 | 5,782.9270329 | 196.4169276 |
| 83 | 124.5961047 | 5,942.2105145 | 200.0997884 |
| 84 | 126.5203840 | 6,103.8499746 | 203.8026391 |
| 85 | 128.4498029 | 6,267.8505832 | 207.5252965 |
| 86 | 130.3843013 | 6,434.2174510 | 211.2675808 |
| 87 | 132.3238206 | 6,602.9556260 | 215.0293157 |
| 88 | 134.2683033 | 6,774.0701012 | 218.8103286 |
| 89 | 136.2176933 | 6,947.5658118 | 222.6104500 |
| 90 | 138.1719358 | 7,123.4476376 | 226.4295137 |
| 91 | 140.1309772 | 7,301.7204043 | 230.2673568 |
| 92 | 142.0947650 | 7,482.3888844 | 234.1238194 |
| 93 | 144.0632480 | 7,665.4577986 | 237.9987445 |
| 94 | 146.0363758 | 7,850.9318169 | 241.8919781 |
| 95 | 148.0140994 | 8,038.8155594 | 245.8033687 |
| 96 | 149.9963707 | 8,229.1135977 | 249.7327590 |
| 97 | 151.9831424 | 8,421.8304560 | 253.6800209 |
| 98 | 153.9743685 | 8,616.9706114 | 257.6450022 |
| 99 | 155.9700037 | 8,814.5384957 | 261.6275620 |
| 100 | 157.9700037 | 9,014.5384957 | 265.6275620 |

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